

A Novel Power Flow Model Based on Linear Programming for Distribution Systems

Abbas Marini ^{1,*}, Josep M. Guerrero ^{2,a}, Luigi Piegari ^{3,†}, and S. S. Mortazavi ^{4,‡}

^{*} Department of Electrical Engineering, Abbaspour College, Shahid Beheshti University, Tehran 1983969411, Iran

^a Department of Energy Technology, Aalborg University, 9220 Aalborg Ø, Denmark.

[†] Department of Electronics, Information & Bio-Engineering, Politecnico di Milano, Milan 20133, Italy

[‡] Department of Electrical Engineering, Shahid Chamran University of Ahvaz, Ahvaz 6135783151, Iran

¹ {Corresponding Author}, Shahid Beheshti University, Abbaspour College, East Vafadar Blvd., Tehranpars, Tehran, Iran. P.O. Box: 16765-1719. Phone: (+98) 21 73932159. Fax: (+98) 21 73932525. Email: a_marini@sbu.ac.ir.

² Email: jos@et.aau.dk

³ Email: luig.piegari@polimi.it

⁴ Email: mortazavi_s@scu.ac.ir

Abstract—Accurate, fast and robust power flow (PF) algorithms are important in distribution systems (DSs) due to their great importance on management of DS applications. From short term to long term horizons, PF equations should be considered into the models. In this paper, a linear programming model is proposed for PF calculations of DSs. To develop the model, some incidence matrices are proposed for modeling DSs based on graph theory. Moreover, a new linear ZIP load model is proposed based on small variations of voltage phase angle in DSs. The linear ZIP load model along with incidence matrices, lead to developing a linear programming model for PF problem. The proposed PF algorithm may easily handle distributed generation, meshed topologies, and unbalanced networks. Since the developed model is linear, it could save in computational complexity and execution time of PF calculations. Furthermore, proposed linear PF constraints could be used in many system studies a structural PF model for DSs. Simulation cases are provided to show the accuracy, robustness, and performance of the proposed PF algorithm.

Index Terms—distribution power flow, linear programming, ZIP load model, graph theory, distributed generation.

I. INTRODUCTION

THE modern smart distribution systems (DSs) require analytical tools providing them decision support in real-time management of system performance. Among them, power flow (PF) calculation is the most important analysis in DSs. Many of computations like network reconfiguration, state estimation, and volt-VAR optimization are based on accurate PF results [1,2]. On the other hand, DSs are widely affected by the new paradigms caused by smart grid (SG) concepts [2]. DSs are going to utilize more meshed topology and to operate as an active network utilizing widespread distributed generations (DGs), with bidirectional flows of power. Moreover, PF models should be capable to efficiently handle topology changes imposed by network reconfigurations. Hence, it seems that robust and efficient PF methods are vital for reliable real-time management of modern DS.

Nevertheless, almost all PF algorithms are nonlinear and require iterations based solutions. The main nonlinearity source is for voltage and current relationship of general ZIP load models. However, PF solutions should be obtained by an iteration procedure. This process is time-consuming and sometime may be diverged in case of ill-conditioning systems. Hence, this may restrict the reliable application of PF in DS management. In real-time analysis and optimization of DS, computation time and robustness become important factors and analytic solutions would be more convenient. An important motivation of this paper is to propose a fast and high-accurate PF algorithm originally developed for DSs based on linear programming approach. Hence, time-consuming procedures, such as Lower Upper (LU) factorization and forward/backward substitution, and ill-conditioned matrices, will not be concerned. All of these advantages lead to the reliable application of PF solution in real-time DS management.

PF methods of DSs may be categorized into Newton–Raphson (NR) [1,2], backward/forward sweep (BFS) [3,4], and direct

methods [5,6]. Some characteristics of DSs like high R/X ratios of lines, mixed overhead-underground cable sections, phase unbalances, and weakly meshed networks, impose some difficulties in applying of conventional PF methods directly for DSs [1,2,5,6]. Usually, NR based PF methods suffer from the ill-conditioning problem and low convergence for DSs. Although some methods were proposed to avoid divergence and ill-conditioning problems [1], it is still a time-consuming process. PF methods based on impedance/admittance matrices, may not efficiently utilize special configuration of DS.

The BFS [5,9], is one of the widely used methods in DSs, since it takes full advantage of weakly meshed topology of DS. In [3] a PF method based on BFS is presented in which, loads are modeled as constant impedance. Generally, PV nodes are considered by a post analysis. However, the convergence of this method deteriorates as the number of PV nodes increases and even maybe diverged [4,7]. Implicit Z-matrix method is another PF method in which loads are modeled as equivalent injections [10]. A PF algorithm is proposed in [11] where network loops are handled by breaking the ties creating some dummy buses. This method is further improved by using graph theory in [12]. But it still needs a post-analysis for PV nodes. A direct method proposed is based on defining two so-called BIBC and BCBV matrices [5] [6]. Actually, these matrices calculate both backward and forward sweeps in a single iteration. However, the direct method is still an iteration based model and could not handle three phase transformers.

There are some linear PF methods in the literature [2,7,8]. A linear PF approximated model for a balanced DS is proposed in [2] which provides an upper and lower bound for voltages of system buses. A linear approximation of ZIP loads is proposed in [7] which makes the PF equations linear. In [8], the Z-matrix method is approximated by linear equations with coefficients based on minimization problem. However, modeling PV nodes is still an important issue in these linear PF methods.

The main idea behind this paper is to propose some PF constraints originally developed for DSs to be applied in optimization models. Common PF constraints are based on nonlinear NR model which have some disadvantages when applied in DSs. On the other hand, BFS and direct PF methods have iterative procedures which limit their application in the optimization model. Hence, in this paper, a novel PF model is proposed for DSs which provides some linear PF constraints. The proposed linear PF constraints could be easily employed in optimization models of DSs from operation to planning horizons. The PF model is then completed by a new linear ZIP load model to establish a linear programming model for the PF problem in DSs. In order to develop the model, some incidence matrices are employed for modeling of DS enable to model meshed and unbalanced networks. Based on the incidence matrices, PF equations are modeled as linear equations. The linear ZIP load model is established based on small variation of the voltage phase angle in DSs which makes load injections as a linear function of bus voltages. The proposed linear PF model leads to elimination of complex matrix procedures and time consuming iteration based solutions. Briefly, paper contributions may be expressed as follows:

- The proposed PF method, takes full advantages of the special structure of DSs based on the graph theory. For radial and meshed DSs, one incidence matrix and two incidence matrices are required for network modeling, respectively. Hence, the minimum number of incidence matrices are employed compared to the other PF methods.
- The topology of DS is only modeled by the incidence matrices and line characteristics and load parameters are modeled in separate matrices. Hence, any topology variation or change in characteristics of the network could be applied shortly into the model. This eliminates any need to recalculation of admittance or impedance matrices. Hence, it could widely reduce execution times for online application of the PF algorithm. This feature could be useful for reconfigurations studies in which DS operator should decide about the optimal status of tie-line switches in the network.
- A new linear ZIP load model is proposed which makes PF algorithm linear. It utilizes small variations of voltage phase angle in DSs to linearize ZIP load model.
- The proposed PF method can handle PV nodes directly into the model and no post sensitivity analysis or in-program iterative reactive power calculation is needed. Also, it may efficiently handle unbalanced networks. This is particularly useful for

studies like state estimation in which a full consideration of unbalances in the DS is needed.

In the rest of the paper, in section II the basis for modeling of DSs is presented. In section III, the proposed ZIP load model is introduced. In section IV, modeling of DG units are presented. Proposed PF linear optimization model is presented in section V. Finally, some conclusions are provided in section VI.

II. MODELING OF DISTRIBUTION SYSTEM

In this section, modeling of DSs is represented which is an extension of our previous work in [13,14] with the same principles.

A. Unbalanced line model

The loading of a distribution feeder is inherently unbalanced due to a large number of unbalanced loads, and un-transposed and multi-phases line segments. Hence, line segments should be modeled using the impedance matrix, Z^{abc} (1). The Z^{abc} includes series and mutual impedances of lines. Consequently, the voltage at sending end and charging current of lines connected to the bus might be calculated as in (2) and (3), respectively [13,14].

$$Z^{abc} = \begin{bmatrix} Z_{aa-n} & Z_{ab-n} & Z_{ac-n} \\ Z_{ba-n} & Z_{bb-n} & Z_{bc-n} \\ Z_{ca-n} & Z_{cb-n} & Z_{cc-n} \end{bmatrix} \quad (1)$$

$$V_i^{abc} = V_j^{abc} + Z_l^{abc} U_l^{abc} \quad (2)$$

$$I_i^{abc,sh} = (1/2) Y_i^{abc} V_i^{abc} \quad (3)$$

Note that U^{abc} just shows the series current of line. Also, $I^{abc,sh}$ shows capacitance current of lines modeled at head buses in π model. The (1)-(3) provide a complete model for voltage and currents of an unbalanced system. For any phase failed to represent, the corresponding row and column in related matrices should be considered as null-entries.

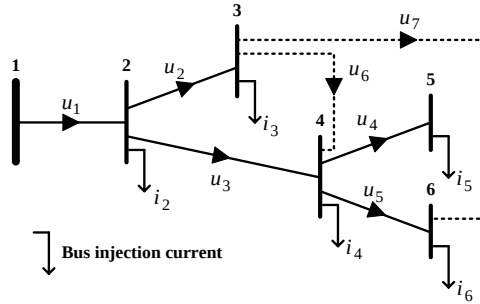


Fig. 1. A typical distribution network

B. Incidence Matrices

Consider a typical distribution network shown in Fig. 1 with an arbitrary tree graph shown with solid lines. Network lines for tree graph are called as "branches" with l index. The number of branches (L) always equals to $N-1$ in a tree graph. Other lines which do not contribute in the tree graph are called as "tie-lines" with m index. The total number of lines including branches and tie-lines are equal to $L+M$.

The loads and all shunt elements in each bus may be converted into an equivalent current injection [5]. For tree graph, by direct application of the KCL, the relationship between branch currents and bus injection currents may be obtained using C matrix in (4a) [14]. The matrix C is called 'branch current to bus current incidence matrix'. The dimension of C is $L \times (N-1)$ and C_{li} equals to one if the current of line l is composed by injection of bus i , as shown in (4b).

$$[U] = [C][I] \quad (4a)$$

$$[C] = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (4b)$$

The relationship between voltage drops of buses and branch currents may also be obtained using transpose of the C [14]. The C^T is of $(N-1) \times L$ dimension which il^{th} element shows the existence of l^{th} line in the route between i^{th} bus and the swing. Hence, the C matrix could be solely used for complete molding of radial DSs. Let the voltage drop of bus i to the swing bus is represented as $\Delta v_i = V_1 - v_i$. Then, the matrix of all voltage drops of system buses could be represented in a matrix form as (5).

$$[\Delta V] = [C]^T [Z][U] \quad (5)$$

Matrix Z is the diagonal matrix of branches impedances and is of $L \times L$ dimension. For unbalanced cases, diagonal elements of Z , should be replaced by the 3×3 matrix of three-phase line impedance shown in (1).

Meshed topologies

Some DS may be energized from both sides due to high voltage drop in long radial lines. In addition, distribution feeders may contain loops created by closing normally open tie-switches. Tie-lines affect the flow of voltage and current in the network. For each tie-line from bus i to bus j carrying current u_m , injection currents of head buses will be changed. Changes in injections of head buses in (4a), could be handled using D matrix [14]. The D is 'branch current to tie-line current incidence matrix' and is of $L \times M$ dimension. New relation of branch currents is shown in (6a) using D matrix (6b). U_m is the vector of tie-line currents. The number of bus injection remains constant in (6a), but M unknown variables of tie-line currents have been added compared to (4a).

$$[U] = [C][I] + [D][U_m] \quad (6a)$$

$$[D] = \begin{bmatrix} 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & -1 \end{bmatrix}^T \quad (6b)$$

Each tie-line constitutes a fundamental loop with some tree branches which make a system of independent equations. Voltage drops in network loops could also be calculated using transpose of D as shown in (7) [14].

$$[0] = [D]^T [Z][U] + [Z_m][U_m] \quad (7)$$

D^T is of $M \times L$ dimension. Also, Z_m is a diagonal matrix of tie-lines impedances represented with m index in (7). M unknown variables in (6a) could be handled using M new equations of (7) for voltage drop in M tie-lines. Hence, network tie-lines could be handled only by another incidence matrix of D . It should be emphasized that bus voltage drops in (5) remain unchanged since routes from buses to the swing don't change. Therefore, (5)-(7) could completely model PF equations for each DS.

III. LINEAR ZIP LOAD MODEL

A well-known load model is the ZIP model in which, each load has been considered consisting of three major parts: a constant power, a constant current and a constant impedance [15,16]. Mathematically, the load current may be determined based on the voltage of load bus and rating condition of the load as (8).

$$i_i^{lo} = \alpha Y^{lo} v_i + \beta I^{lo,i} e^{j(\angle v_i + \angle Y^{lo})} + \gamma (S_o^{lo} / v_i)^* \quad (8)$$

Calculation of i -part and p -part injections of the load currents are nonlinear equations. In this section, a linear ZIP load model is proposed based on small variations of the voltage phase angle.

131 A. z-part injection

132 Injection of z-part is calculated using constant admittance. The load admittance and its phase angle are calculated in (9a) and
133 (9b), respectively. Hence, real and imaginary parts of z-part injection are calculated in (9c) and (9d), respectively.

$$Y = G + jB = S_0^* = P_0 - jQ_0 \quad (9a)$$

$$\angle Y = \theta = \tan^{-1}(B/G) = \tan^{-1}(-Q_0/P_0) \quad (9b)$$

$$i^{z, \text{re}} = P_0 v^{\text{re}} + Q_0 v^{\text{im}} \quad (9c)$$

$$i^{z, \text{im}} = -Q_0 v^{\text{re}} + P_0 v^{\text{im}} \quad (9d)$$

134 B. i-part injection

135 In order to calculate i-part injection of load, fixed power factor criterion of the load should be satisfied. The fixed magnitude is
136 determined based on rating condition. But, the phase angle of the load should be modified according to the phase angle of voltage
137 and admittance (8). Almost all linear ZIP load models have assumed that both magnitude and phase angle of i-part current to be
138 fixed [5,7,16]. However, it may not be a correct assumption based on fixed power factor criterion of the load. The magnitude of i-
139 part injection is determined as (10a). Therefore, real and imaginary parts could be calculated as (10b) and (10c).

$$I^i = |S_0| = \sqrt{P_0^2 + Q_0^2} \quad (10a)$$

$$i^{i, \text{re}} = I^i \cos(\delta + \theta) \quad (10b)$$

$$i^{i, \text{im}} = I^i \sin(\delta + \theta) \quad (10c)$$

140 In DSs, considering the voltage angle of the substation as a reference, the imaginary part of the voltage is smaller than the real
141 part, by several times [15,16]. Hence, the phase angle and magnitude of the imaginary part of voltages of system buses lead to
142 zero. This leads following triangular approximations in (11) to be applicable in DSs.

$$\cos(\delta_i) = 1; \quad \sin(\delta_i) = \delta_i \quad (11a)$$

$$\delta_i = \tan^{-1}(v_i^{\text{im}}/v_i^{\text{re}}) \approx v_i^{\text{im}}/v_i^{\text{re}} \approx v_i^{\text{im}} \quad (11b)$$

143 Utilizing cosines and sinus of the sum of two angles to calculate (10) and applying approximations of (11), imaginary and real
144 parts may be calculated as (12a)- (12b), respectively.

$$i^{i, \text{re}} = I^i (\cos(\theta) - v^{\text{im}} \sin(\theta)) \quad (12a)$$

$$i^{i, \text{im}} = I^i (v^{\text{im}} \cos(\theta) + \sin(\theta)) \quad (12b)$$

145 C. p-part injection

146 The p-part injection is a nonlinear function of voltage (8). Taylor series is a well-known method to linearize the nonlinear
147 equations around any point of interest. The function of $1/v$ is analytical around unity and may be approximated with $2-v$. Hence,
148 the injection of p-part could be calculated as (13a). Using some simple calculations, real and imaginary parts of the p-part injection,
149 could be represented as (13b)-(13c), respectively.

$$i^p = (S_0/v_i)^* = (P_0 - jQ_0)(2 - v_i)^* \quad (13a)$$

$$i^{p, \text{re}} = P_0 (2 - v^{\text{re}}) + Q_0 v^{\text{im}} \quad (13b)$$

$$i^{p, \text{im}} = P_0 v^{\text{im}} - Q_0 (2 - v^{\text{re}}) \quad (13c)$$

Collecting all expressions for real and imaginary parts of currents in (9), (12) and (13), the real and imaginary parts of current can be calculated as (15a)-(15b) by k coefficients (15).

$$i_i^{\text{lo, re}} = k_1^{\text{lo}} v_i^{\text{re}} + k_2^{\text{lo}} v_i^{\text{im}} + k_3^{\text{lo}} \quad (14a)$$

$$i_i^{\text{lo, im}} = k_4^{\text{lo}} v_i^{\text{re}} + k_5^{\text{lo}} v_i^{\text{im}} + k_6^{\text{lo}} \quad (14b)$$

Coefficients of proposed ZIP load model are shown with $k_1, \dots, k_6$ in (15). The k coefficients are constant numbers which can be calculated in offline analysis. The model of (14) is linear which makes the PF equations linear.

$$k_1 = (\alpha - \gamma) P_0 \quad (15a)$$

$$k_2 = (\alpha - \gamma) Q_0 - \beta I^i \sin(\theta) \quad (15b)$$

$$k_3 = \beta I^i \cos(\theta) + 2\gamma P_0 \quad (15c)$$

$$k_4 = (\gamma - \alpha) Q_0 \quad (15d)$$

$$k_5 = (\alpha - \gamma) P_0 + \beta I^i \sin(\theta) \quad (15e)$$

$$k_6 = \beta I^i \sin(\theta) - 2\gamma Q_0 \quad (15f)$$

IV. MODELING OF DISTRIBUTED GENERATION

Distributed generations are becoming more prevalent in the power industry. They include a broad variety of technologies such as wind turbines, photovoltaic systems, fuel cells, micro-turbines and different types of energy storage systems. DGs are more connected to the network in distribution levels. Hence, it is necessary to provide analytical tools for DGs in DSs.

In general, the energy source of DG units may be categorized into determined and undetermined energy sources. Besides, they may be connected to the grid via synchronous generators, induction generators, and power electronic interfaces. Various energy sources combined with different energy converters result in various models. DG units could be modeled either as a PQ bus or PV bus in PF studies [6,17,18]. Some DG models are shown in Table I [6,19].

TABLE I- DG MODELS FOR POWER FLOW STUDIES

Energy source	Energy converter	Model
micro-turbine	synchronous gen.	PQ
gas turbine	synchronous gen.	PV
wind	Induction gen.	PV
Fuel-cell	Inverter	PQ
photovoltaic	Inverter	PQ

For a DG modeled as PQ bus, injection of DG may be calculated using (14). The voltage of DG modeled as PV bus could be calculated as (16) in which approximations of (11) is applied.

$$v_i^{\text{pv}} = |V_i^{\text{pv}}| \cos(\delta_i) + j |V_i^{\text{pv}}| \sin(\delta_i) \approx V_i^{\text{pv}} + j V_i^{\text{pv}} \delta_i \quad (16)$$

Injection of the DG unit could be determined as (17a). By some simple calculations, real and imaginary parts injections of DG could be represented as (17b)-(17c), respectively.

$$i_i^{\text{pv}} = \frac{S_i^{\text{pv}*}}{V_i^{\text{pv}*}} = (P_i^{\text{pv}} - j q_i^{\text{pv}}) (|V_i^{\text{pv}}| + j |V_i^{\text{pv}}| \delta) / |V_i^{\text{pv}}|^2 \quad (17a)$$

$$i_i^{\text{pv, re}} = (P_i^{\text{pv}} + q_i^{\text{pv}} \delta) / |V_i^{\text{pv}}| \quad (17b)$$

$$i_i^{\text{pv, re}} = (P_i^{\text{pv}} \delta - q_i^{\text{pv}}) / |V_i^{\text{pv}}| \quad (17c)$$

The developed model in (16)-(17) may completely handle a DG unit modeled as a PV bus. Note that the second term in the numerator of (17b) makes the equation nonlinear. But, due to low amounts of phase angle in DSs, one may ignore the terms associated with δ in (17b).

V. LINEAR PROGRAMMING MODEL FOR POWER FLOW

A PF model based on linear programming is proposed here using the developed models. Although using an objective function (OF) in PF problem seems to be needless; but, notable application of optimization models in system studies makes it justifiable. The OF should be defined based on the type of the problem, while the proposed linear constraints could be used as efficient PF equations developed for DSs. PF constraints based on admittance matrix or Newton-Raphson models suffer from poor numerical instability and slow convergence rate for DSs. However, the proposed PF constraints are defined based on DSs and could be efficient in the convergence of the model.

A. Injection current of a bus

The bus injection current consists of three major parts; injections due to connected load modeled in (14), injections due to shunt capacitance of lines connected to bus based on π model presented in (3), and injections due to capacitor bank modeled as constant susceptance, B^{cp} [15]. Collecting all injection terms, total injection current could be calculated as (18).

$$i_i = i_i^{\text{lo}} + j (B_i^{\text{sh}} + B_i^{\text{cp}}) v_i \quad (18)$$

The (18) is just valid for PQ load buses with specified active and reactive powers. For PV buses, the injection current may be determined using (17b) and (17c).

B. Optimization model

The complete proposed PF optimization model is represented in (19). It consists of a problem-dependent OF, exact constraints of PF equations, approximated constraints of load model, and of PV buses. The OF in (19a) is defined as the minimization of summation of bus voltage drops. However, it could be defined differently, since it does not affect the validity of the PF model.

$$\text{OF} : \min \left\{ \sum_{i=2}^V (V_1^{\text{re}} - v_i^{\text{re}}) + (V_1^{\text{im}} - v_i^{\text{im}}) \right\} \quad (19a)$$

s.t. linear exact PF constraints:

$$u_l^{\text{re}} = \sum_i C_{li} i_i^{\text{re}} + \sum_m D_{lm} u_m^{\text{re}}, \forall l, \forall i \quad (19b)$$

$$u_l^{\text{im}} = \sum_i C_{li} i_i^{\text{im}} + \sum_m D_{lm} u_m^{\text{im}}, \forall l, \forall i \quad (19c)$$

$$V_1^{\text{re}} - v_i^{\text{re}} = \sum_l C_{il}^T \{ R_l u_l^{\text{re}} - X_l u_l^{\text{im}} \}, \forall i, \forall l \quad (19d)$$

$$V_1^{\text{im}} - v_i^{\text{im}} = \sum_l C_{il}^T \{ R_l u_l^{\text{im}} + X_l u_l^{\text{re}} \}, \forall i, \forall l \quad (19e)$$

$$0 = \sum_l D_{ml}^T \{ R_l u_l^{\text{re}} - X_l u_l^{\text{im}} \} + R_m u_m^{\text{re}} - X_m u_m^{\text{im}}, \forall m \quad (19f)$$

$$0 = \sum_l D_{ml}^T \{ R_l u_l^{\text{im}} + X_l u_l^{\text{re}} \} + R_m u_m^{\text{im}} + X_m u_m^{\text{re}}, \forall m \quad (19g)$$

s.t. linear approximated load (PQ buses) constraints:

$$i_i^{\text{re}} = k_1 v_i^{\text{re}} + (k_2 - B_i^{\text{sh}} - B_i^{\text{cp}}) v_i^{\text{im}} + k_3, \forall i \in \Omega^{\text{pq}} \quad (19h)$$

$$i_i^{\text{im}} = (k_4 + B_i^{\text{sh}} + B_i^{\text{cp}})v_i^{\text{re}} + k_5 v_i^{\text{im}} + k_6, \forall i \in \Omega^{\text{pq}} \quad (19\text{i})$$

s.t. linear approximated PV buses constraints:

$$i_i^{\text{re}} = P_i^{\text{pv}} / V_i^{\text{pv}}, \forall i \in \Omega^{\text{pv}} \quad (19\text{j})$$

$$i_i^{\text{im}} = (P_i^{\text{pv}} \delta - q_i^{\text{pv}}) / V_i^{\text{pv}}, \forall i \in \Omega^{\text{pv}} \quad (19\text{k})$$

$$v_i^{\text{re}} = V_i^{\text{pv}}, \forall i \in \Omega^{\text{pv}} \quad (19\text{l})$$

$$v_i^{\text{im}} = V_i^{\text{pv}} \delta, \forall i \in \Omega^{\text{pv}} \quad (19\text{m})$$

The constraints of (19b)-(19g) represent PF equations of the network based on models developed in (5), (6a), and (7). Since no approximations were used in moldering DS, these constraints could exactly model PF equations. Hence, they could be used as an efficient alternative to conventional Newton-Raphson or admittance matrix models for PF equations. The constraints of (19b)-(19c) relate the line currents to bus injection and tie-lines currents based on (6a). The equations of (19d)-(19e) represent the voltage drop of system buses with respect to the voltage of the swing bus developed in (5). The constraints of (19f)-(19g) express voltage drop in loops created by tie-lines based on (7). For radial DSs, the (19f)-(19g) and the terms associated with D matrix in (19b)-(19c), should not be considered into the model.

Injection currents of PQ buses are modeled in (19h)-(19i) based on (14) and (18). Since the exact injection of a ZIP load in (8) was linearized in (14), some approximations are involved in these constraints. However, one could use the exact ZIP load model which result in a nonlinear PF model. It could be solved iteratively or using a nonlinear optimization model. In simulations, errors associated with linearized models will be analyzed. Note that, injections of DGs modeled as PQ should also be modeled using (19h)-(19i).

The constraints of (19j)-(19m) are proposed for PV buses. The (19j)-(19k), represent the injection of PV buses based on (17). Besides, the (19l)-(19m) define real and imaginary parts of voltage based on (16), respectively. Limits of reactive power in PV buses are ensured by a post-analysis. After convergence of the problem, the limitations of reactive power will be analyzed for PV buses. In case of any violation, the PV bus is turned to a PQ bus and the reactive injection power is adjusted to marginal values. Then, the PF model will be resolved.

VI. SIMULATION RESULTS

The proposed models are validated by various simulations. Linear optimization models are simulated in GAMS [20] and typical nonlinear algorithms are modeled in MATLAB. The ZIP coefficients are assumed as $\alpha = 0.025$, $\beta = 0.51$, $\gamma = 0.465$ [16]. All models are implemented in a personal PC equipped with 2.93 GHz processor and 4 GB RAM.

A. Proposed Load Model

In order to illustrate the performance of the proposed linear ZIP load model, it was implemented in a typical PF problem. Three different linear ZIP load models were compared with the exact nonlinear model as follows:

- *Model A*: Nonlinear exact ZIP load model of (8).
- *Model B*: The linear model of [7] in which just the p -part term is linearized and the phase angle of i -part is not modified.
- *Model C*: The linear model of [16] in which the ZIP model is approximated by a ZI model with new coefficients.
- *Model D*: The proposed load model which both nonlinear terms of ZIP model were approximated.

The IEEE-18 bus test system was used for the comparison of load models [13]. The system is a radial distribution system, containing 16 busses at 12.5 kV, and 2 busses at 138 kV. The error in the calculation of voltage magnitude in different load models is shown in Fig. 2. The least error is for the proposed ZIP model. The error is typically less than 0.002 pu and 0.02 degree for voltage magnitude and phase angle, respectively. The ZI load model shows a relatively high error. This drawback may arise from

coefficients of ZI model calculated for fixed conditions. This may restrict the application of the model for changing conditions. Even though both models of B and D are linear, but the proposed ZIP model shows better accuracy.

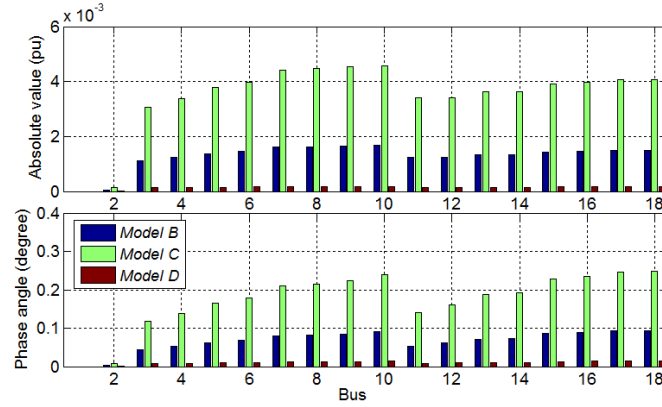


Fig. 2. Calculation error of voltage in different ZIP load models

The accuracy of the load model depends on the type of loads. In order to show the capability of the proposed ZIP model, the coefficients of β and γ were changed from 0 to 1 in steps of 0.2. In each combination of loads, the maximum magnitude error of voltage for load models of B and D was calculated. Calculation errors are shown in Fig. 3. As it could be seen, the calculation error of the proposed load model is less in all combination of loads. Also, since the constant current part of the load does not modify in load model of B , the error tends to increase with the increase in β . Hence, the proposed load model may be utilized as a reliable linear load model for DSs in which constant current loads are dominant.

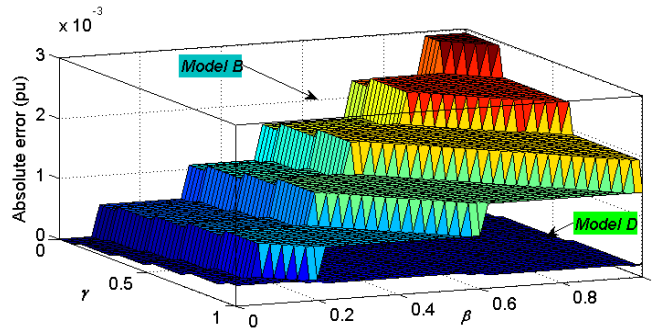


Fig. 3. Calculation error of voltage in different ZIP load models

Accuracy of linear ZIP load models is proportional to voltage drop. The more voltage drop, the less accuracy of load model. The accuracy of the proposed ZIP model is also examined in various voltage drops. In Fig. 4, the maximum calculation error is shown. Since the term of $1/v$ was approximated around the unity, calculation error is increased with an increase in voltage drop. The model is accurate enough to 0.2 pu voltage drop. Hence, the calculation error of the model D may be less enough in typical operation of DS, since almost all distribution operators avoid a voltage drop of 10%. The proposed linear ZIP model may bring a good compromise between accuracy and performance.

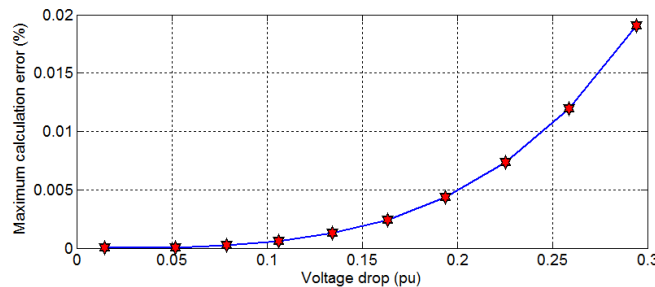


Fig. 4. Maximum calculation error versus maximum voltage drop

The proposed PF optimization model was implemented in some test systems. PF outputs are compared with the direct PF method of [5]. Utilized PF methods are as:

- *M1*: proposed linear optimization model.
- *M2*: direct PF method of [5].

1) Eight-bus unbalanced network

An 8-bus unbalanced distribution feeder [21] shown in Fig. 5 is used to show the capability of the proposed PF model for unbalanced cases. Table II represents PF outputs for different methods. The highest error is in the order of 10^{-4} shows the accuracy of the proposed method compared to the direct method. In another simulation, the proposed ZIP linear load model was utilized into the direct PF method of *M2*. Its results are as the same with results of *M1*. Low errors are due to the approximations of linear ZIP load model and PF constraints are accurate as BIBC and BCBV matrices. In the last two rows of Table III, execution time and the number of iterations are represented. Execution times for the *M2* and *M1* are 4.37 ms and 0.671 ms, respectively. The proposed method is more than six times faster than the BIBC method for this case. Also, the solution is obtained with no iteration, whereas the *M2* needs 6 iterations for a tolerance of 10^{-6} . The proposed linear optimization model eliminates the need for the iterative solution. This may lead to a great saving in execution time to be implemented in real cases in near real-time applications.

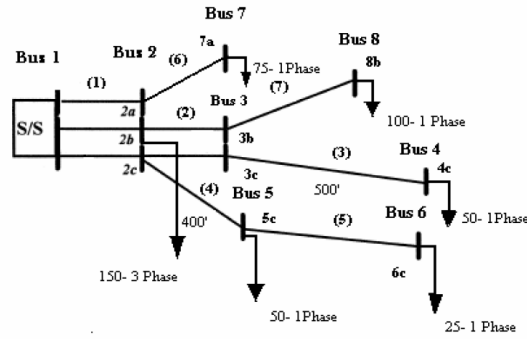


Fig. 5. Eight-bus unbalanced test network

TABLE II- RESULTS OF THE 8-BUS UNBALANCED SYSTEM

PF method		<i>M2</i>		<i>M1</i>	
Load model		ZIP		linear ZIP	
Bus	Phase	v	δ	v	δ
2	a	0.9977	-0.0088	0.9977	-0.0088
	b	0.9939	-2.0903	0.9939	-2.0903
	c	0.9734	2.0968	0.9735	2.0968
3	b	0.9820	-2.0823	0.9820	-2.0824
	c	0.9502	2.0930	0.9503	2.0930
4	c	0.9481	2.0931	0.9482	2.0931
5	c	0.9699	2.0969	0.9700	2.0969
6	c	0.9676	2.0970	0.9677	2.0970
7	a	0.9953	-0.0096	0.9953	-0.0096
8	b	0.9787	-0.0088	0.9786	-0.0088
Running time (ms)		4.37		0.671	
Iterations		6		1	

2) IEEE 33-bus test feeder

The PF methods of *M1* and *M2* are also implemented on the IEEE 33-bus test system [22]. The test system consists of 32 branches and 5 tie-switches. At first, tie-switches were ignored. The voltage profile using *M1* is shown in Fig. 6. Results of applying *M2* are almost the same as *M1* and the highest error is 0.0002 pu for bus #18. The simulations are performed in 2.981 ms and 0.781

ms respectively for $M1$ and $M2$. It shows the performance of the proposed PF model for the non-iterative solution of the problem. The OF in the first case is equal to 1.5909 which shows an average 4.7% voltage drop in each bus. The least voltage is 0.918 pu for bus #18. Tie-switches was considered in another case whose results are also shown in Fig. 6. Improvement of voltage profile is obtained considering tie-switches. The OF is equal to 0.9286 which shows a decrease in the voltage drop across the network. The average voltage drop is reduced to 2.7%.

The capability of the proposed PF method for handling DGs was investigated in another simulation. Two DG units worked as PV buses are placed in bus #18 and #33. They wish to adjust the voltage at 1 pu. The rated active powers are equal to 0.035 pu and 0.1 pu, respectively. At first, tie-switches were ignored in simulations. Voltage profile is presented in Fig. 6. It could be seen that the proposed method can successfully adjust the voltage for PV nodes at 1 pu. Reactive power injections are equal to 0.049 pu and 0.067 pu for DGs in bus #18 and #33, respectively. The OF is equal to 0.5109 which shows a decrease compared to the previous cases including no DG. It shows an average 1.5% voltage drop in each bus. Finally, tie-switches and DGs were both considered. It's expected that voltage profile to be even better than previous cases. The voltage profile is shown in Fig. 6. It shows more decrease in voltage drop compared to the case with no tie-switches. The OF, in this case, is equal to 0.4215 which shows an average voltage drop of 1.2%.

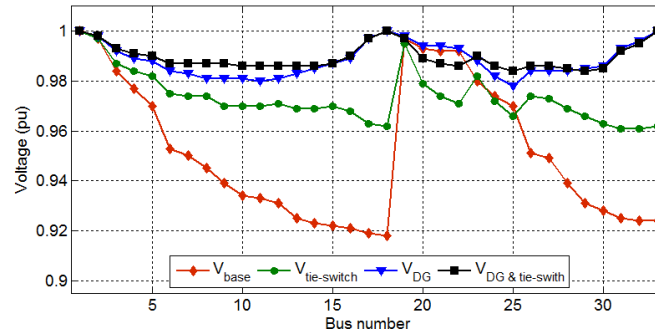


Fig. 6. Voltage profile of IEEE 33-bus system

An important issue that may lead the PF algorithms to be diverged, is high R/X ratios. In order to show the robustness of the method $M1$, simulations were repeated for different R/X ratios. In all simulation cases, the PF algorithm successfully converged. For high R/X ratios, the error increases since load model approximations become less accurate for higher voltage drops. Another problem which may diverge PF algorithm is ill-conditioned equations. It usually occurs when the network contains some very short lines or long lines. Some impedances are multiplied by ten and some others are divided by ten in another simulation. In all cases, the convergence of the proposed PF method is observed.

3) IEEE 123-bus test feeder

In order to show the capability of the model for handling large unbalanced networks, it was implemented on the modified IEEE 123-bus network [22]. Outputs of PF are shown in Fig. 7. The bus voltage, especially for phase 'a', is somewhere below the desired level of system voltage. The OF, in this case, is equal to 11.3151. In the second case, a DG unit was placed at bus #95. The rated power of DG is equal to 0.25 pu working in PV mode. Voltage profile in the new case is also shown in Fig. 7. Improving the voltage profile as well as adjusting the voltage magnitude of bus #95 is obviously observed in Fig. 7. The OF is equal to 5.6948 which shows a decrease in voltage drop in the network. Also, the DGs inject reactive power of 4.582 pu, 3.056 pu, and 4.115 pu into the network for each 'abc' phases, respectively. Running time is 8.32 ms and 1.15 ms for methods of $M1$ and $M2$, respectively. Low running time with no-iteration procedure again reveal the performance of the proposed PF model.

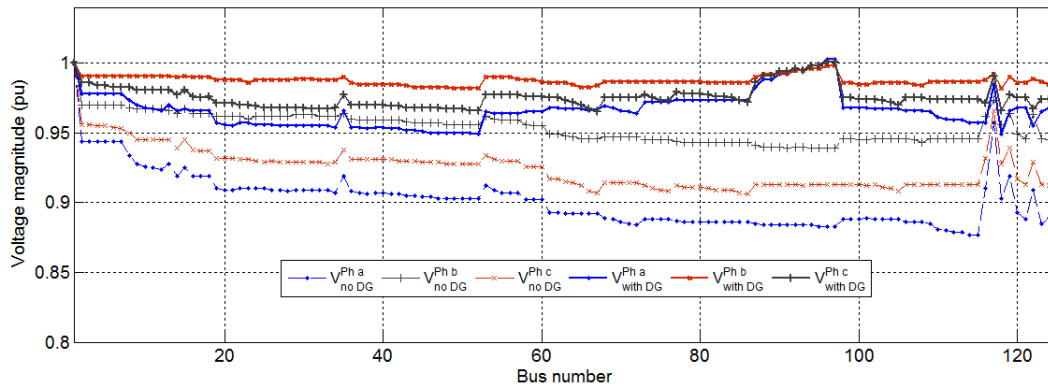


Fig. 7. Voltage profile of IEEE 123-bus system with and without DGs

Other DS studies could utilize linear PF models as well. Volt-VAR optimization, expansion studies, harmonic PF, and filter placement are of such complexed-model studies. For instance, reconfiguration is a mixed-integer nonlinear important problem which should be done in near real-time. Heuristic methods suffer from an un-guaranteed global optimal solution. The proposed linear PF constraints could be used for utilizing speed and robustness of linear optimization models. A major advantage of linear optimization models is reaching global optima in a polynomial acceptable time.

VII. CONCLUSION

In this paper, a linear programming approach was proposed for PF in DSs based on some proposed incidence matrices. Incidence matrices were defined based on graph theory. They could completely model DSs and could easily handle any meshed topology. Furthermore, based on small variation of voltage phase angle in DS, a linear ZIP load model was proposed which makes PF algorithm linear. Finally, a linear optimization model was proposed for PF calculations.

The proposed PF model was implemented in some distribution test systems. The results showed the robustness, accuracy, and performance of the model. Time-consuming processes such as FBS is not needed. Furthermore, topology-based incidence matrices lead to the elimination of time-consuming procedures, such as LU factorization and also ill-conditioned matrices. All of these advantages lead to the reliable application of proposed PF algorithm in DS analysis. It can provide decision support for real-time optimization in fundamental and harmonic analysis of DS problems such as state estimation, DS reconfiguration, Volt-VAR optimization, harmonic analysis, power quality studies, and reactive power procurement.

NOMENCLATURE

In the paper, single parameters, single variables, and matrices of variables and parameters are shown with capital letters, small letters, and capital letters in brackets, respectively as follows:

Sets

i, j	Network buses.
l	Network branches (tree graph)
m	Network tie-lines

superscripts

im	The imaginary part of a complex number
lo	Load
re	The real part of a complex number
sh	Shunt capacitance of line in pi model
z, i, p	ZIP parts of voltage dependent load

Parameters

C	Branch current to bus injection incidence matrix.
D	Bus voltage to branch current incidence matrix.
$Z = R+jX$	Line impedance (branches and tie-lines)
P	Active power.
Q	Reactive power.
$Y = G+jB$	Parallel admittance of system buses.
k	Load coefficients in proposed ZIP load model.
α, β, γ	Coefficients of ZIP loads
δ	The phase angle of the voltage
θ	The phase angle of load admittance

Variables

U	Branch currents.
V	Bus voltages.
I	Bus injection currents.

REFERENCES

- [1] O. L. Tortelli, E. M. Lourenço, A. V. Garcia, and B. C. Pal, "Fast decoupled power flow to emerging distribution systems via complex pu normalization," *IEEE Trans. on Pow. Syst.*, vol. 30, no. 3, pp. 1351-1358, 2015.
- [2] S. Bolognani and S. Zampieri, "On the existence and linear approximation of the power flow solution in power distribution networks," *IEEE Trans. on Pow. Syst.*, vol. 31, no. 1, pp. 163-172, 2016.
- [3] A. Augugliaro, L. Dusonchet, S. Favuzza, M. G. Ippolito, and E. R. Sanseverino, "A backward sweep method for power flow solution in distribution networks," *Int. Jou. of Elec. Pow. & Ener. Syst.*, vol. 32, no. 4, pp. 271-280, 2010.
- [4] Y. Ju, W. Wu, B. Zhang, and H. Sun, "An extension of FBS three-phase power flow for handling PV nodes in active distribution networks," *IEEE Trans. on Sma. Gri.*, vol. 5, no. 4, pp. 1547-1555, 2014.
- [5] J.-H. Teng, "A direct approach for distribution system load flow solutions," *IEEE Trans. on Pow. Deliv.*, vol. 18, no. 3, pp. 882-887, 2003.
- [6] J.-H. Teng, "Modelling distributed generations in three-phase distribution load flow," *IET Gener., Transm. & Distr.*, vol. 2, no. 3, pp. 330-340, 2008.
- [7] A. Garces, "A Linear Three-Phase Load Flow for Power Distribution Systems," *IEEE Trans. on Pow. Syst.*, vol. 31, no. 1, pp. 827-828, 2016.
- [8] H. Ahmadi, J. R. Marti, and A. von Meier, "A Linear Power Flow Formulation for Three-Phase Distribution Systems," *IEEE Trans. on Pow. Syst.*, 2016.
- [9] D. Shirmohammadi, H. Hong, A. Semlyen, and G. Luo, "A compensation-based power flow method for weakly meshed distribution and transmission networks," *IEEE Trans. on Pow. Syst.*, vol. 3, no. 2, pp. 753-762, 1988.
- [10] T.-H. Chen, M.-S. Chen, K.-J. Hwang, P. Kotas, and E. A. Chebli, "Distribution system power flow analysis-a rigid approach," *IEEE Trans. on Pow. Deliv.*, vol. 6, no. 3, pp. 1146-1152, 1991.
- [11] M. Haque, "Efficient load flow method for distribution systems with radial or mesh configuration," *IEE Proc.-Gener., Transm. & Distr.*, vol. 143, no. 1, pp. 33-38, 1996.

- [12] N.-C. Yang, "Three-phase power flow calculations using direct Z BUS method for large-scale unbalanced distribution networks," *IET Gener., Transm. & Distr.*, vol. 10, no. 4, pp. 1048-1055, 2016.
- [13] L. P. Abbas Marini, S. S. Mortazavi, M. S. Ghazizadeh, "A linear programming approach to distribution Power Flow," presented at the 6th International Conference on Clean Electrical Power (ICCEP), Santa Margherita, Italy, 2017.
- [14] A. Marini, S. Mortazavi, L. Piegari, and M.-S. Ghazizadeh, "An efficient graph-based power flow algorithm for electrical distribution systems with a comprehensive modeling of distributed generations," *Electric Power Systems Research*, vol. 170, pp. 229-243, 2019.
- [15] W. H. Kersting, *Distribution system modeling and analysis*. CRC press, 2012.
- [16] J. R. Martí, H. Ahmadi, and L. Bashualdo, "Linear power-flow formulation based on a voltage-dependent load model," *IEEE Trans. on Pow. Deliv.*, vol. 28, no. 3, pp. 1682-1690, 2013.
- [17] C. S. Cheng and D. Shirmohammadi, "A three-phase power flow method for real-time distribution system analysis," *IEEE Trans. on Pow. Syst.*, vol. 10, no. 2, pp. 671-679, 1995.
- [18] H. Chen, J. Chen, D. Shi, and X. Duan, "Power flow study and voltage stability analysis for distribution systems with distributed generation," in *IEEE Power Engineering Society General Meeting*, 2006, p. 8 pp.: IEEE.
- [19] S. Moghaddas-Tafreshi and E. Mashhour, "Distributed generation modeling for power flow studies and a three-phase unbalanced power flow solution for radial distribution systems considering distributed generation," *Elec. Pow. Syst. Rese.*, vol. 79, no. 4, pp. 680-686, 2009.
- [20] E. Rosenthal, "GAMS-A user's guide," in *GAMS Development Corporation*, 2008: Citeseer.
- [21] B. a. M. Naik, Nikhil and Behera, Somanath, "Load flow analysis of unbalanced radial distribution systems," BTech, Department of Electrical Engineering, NIT Rourkela 2014.
- [22] I. P. D. S. A. Subcommittee's. (2013). *Distribution Test Feeders*. Available: <http://ewh.ieee.org/soc/pes/dsacom/testfeeders/index.html>.